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Some two-dimensional internal waves in a stratified fluid

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Experiments are presented in which two-dimensional internal waves were generated in a stably stratified fluid by the movement of a long horizontal circular cylinder. The cylinder was moved with a constant velocity at Reynolds numbers based on the diameter of the cylinder of between 6 and 100. Under these conditions the internal wave system is stationary with respect to the moving cylinder and the phase configuration of the waves compares well with Lighthill's theory for waves in dispersive systems.

1. Introduction

A body moving in a stable density stratified fluid may set up an internal wave system. Görtler (1943) and Mowbray & Rarity (1967*a*) have compared the wave system developed by an oscillating body in such a fluid with linearized theories. Görtler studied the steady-state wave systems by means of a shadowgraph technique and compared the results with a theory based on the characteristics of the linear equations, while Mowbray & Rarity used the Toepler–Schlieren system to look at the waves and used a theory based on group velocity arguments. Lighthill (1967) and Mowbray & Rarity (1967*b*) have considered theories for the internal wave system behind a vertically moving sphere. In all cases the agreement between the linear theory and the observations is very good.

This paper presents some experimental results which show the phase configuration of two-dimensional internal waves generated by a body moving in a stably stratified fluid. The body was a long circular cylinder moving with a constant velocity normal to its horizontal longitudinal axis so that its path made an angle with the horizontal. Lighthill (1967) gave a general theory for the waves generated in dispersive systems by travelling disturbances and it will be shown how this theory predicts the phase configuration of the internal waves around the cylinder.

2. Theoretical predictions

The application of Lighthill's theory to this problem is presented by Rarity (1967). The equation of motion for an inviscid two-dimensional flow takes the form

$$\frac{D^2}{Dt^2}\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial x^2}\right) = -\omega_0^2\frac{\partial^2\psi}{\partial x^2}, \tag{1}$$

where

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x} - V\frac{\partial}{\partial y}\right)$$

and x and y are the horizontal and vertical co-ordinates relative to a body whose constant velocity relative to the undisturbed fluid is (U, V). ψ is the stream function defined by

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}, \quad \text{and} \quad \omega_0^2 = -g\rho_0^{-1}\frac{\partial \rho_0}{\partial y}$$

is the square of the Väisälä–Brunt frequency which we shall consider constant. ρ_0 is the undisturbed density and g is the acceleration due to gravity. The Boussinesq approximation has been used, that is, the density changes have only been retained in the buoyancy term. Rarity does not use this approximation and a further term is present throughout his analysis. The extra term, which is of the form $W\omega_0/2g$ where $W\,[= (U^2+V^2)^{\frac{1}{2}}]$ is the velocity of the body, changes the lines of constant phase and alters the asymptote of the waves ahead of the body. However, the predicted change in angle of the asymptote is less than 10^{-3} radians in these experiments and it is reasonable to neglect the extra term.

We look for a plane wave solution of the form

$$\psi = \psi_0 \exp i(k_1 x + k_2 y - \omega t) \tag{2}$$

so that the dispersion relation obtained from (1) takes the form

$$P(\omega + \mathbf{U}.\mathbf{k},\ k_1, k_2) = (k_1^2 + k_2^2)(\omega + Uk_1 + Vk_2)^2 - \omega_0^2 k_1^2 = 0. \tag{3}$$

Lighthill (1967) discusses the effects of forcing terms moving with a constant velocity. The forcing term is assumed to vanish outside a limited region around the origin and it is expressed as a Fourier integral in wave-number space. Using the above equations with a forcing term present allows a formal solution for ψ to be written as an integral over all wave-numbers. The waves that exist in a certain direction from the forcing region are those with wave-numbers corresponding to points on the wave-number surface which have normals, drawn towards higher ω, which point in that particular direction.

Curves of constant phase are represented by $\mathbf{k}.\mathbf{r} = A$ where A is a constant and $\mathbf{r}$ represents (x, y). Now $\mathbf{r}$ is parallel to the normal to P at $\mathbf{k}$ and is therefore a simple multiple of ∇P where ∇ is the gradient operator in $\mathbf{k}$ space. Thus the locus of points of constant phase is given by $A\nabla P/(\mathbf{k}.\nabla P)$ which is evaluated from (3) and may be written in the parametric form

$$\frac{AV}{\omega_0}\left(\frac{G^3\cot\alpha - 1}{G^2(G^2+1)^{\frac{1}{2}}},\quad \frac{G\cot\alpha + 2 + G^2}{G(G^2+1)^{\frac{1}{2}}}\right), \tag{4}$$

where $G = k_1/k_2$ and α is the angle which the path of the body makes with the horizontal. Curves of constant phase are plotted in terms of $\omega_0 x/AV$ and $\omega_0 y/AV$ for various values of α in figures 1–4. The families of curves in these figures give an impression of the wave system but the distances between the curves do not represent any particular change of phase.

3. The experiments

The water tank used in the experiments was 160 cm long, 90 cm high and 55 cm from front to back. A stratified salt solution having a linear variation of density with depth was obtained by the method described by Mowbray (1966). A long circular cylinder, 0·94 cm diameter, was suspended in the solution by two vertical supports from a trolley which ran with constant velocity on adjustable rails above the tank. In all cases the longitudinal axis of the cylinder remained horizontal and the values of α used were 90°, 45°, 20° and 10°. The motion was observed through the glass sides of the tank with a schlieren system developed and de-

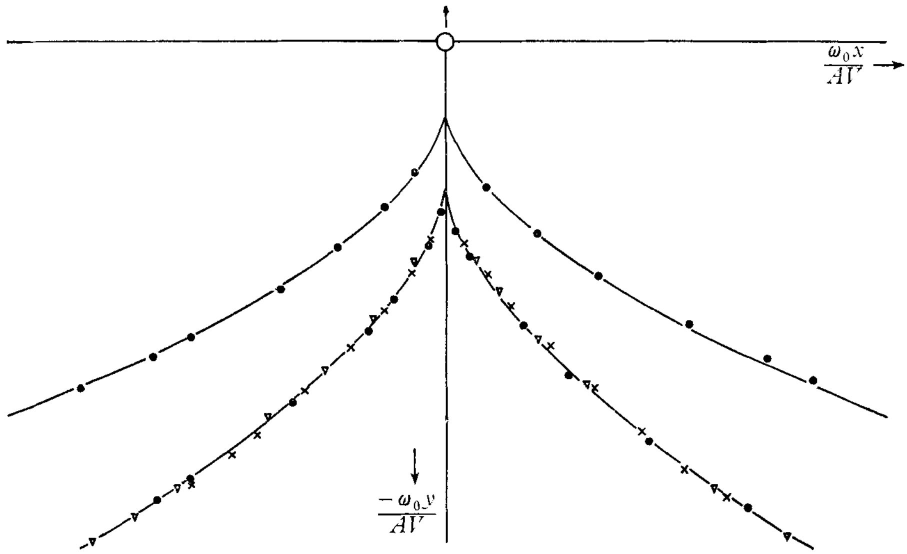

FIGURE 1. Curves of constant phase when the cylinder moves in a vertical direction. Experimental points are taken from photographs of the wave system around the 0·94 cm diameter cylinder moving with velocities of: ●, 0·38 cm/s; ×, 0·56 cm/s; ▽, 0·57 cm/s.

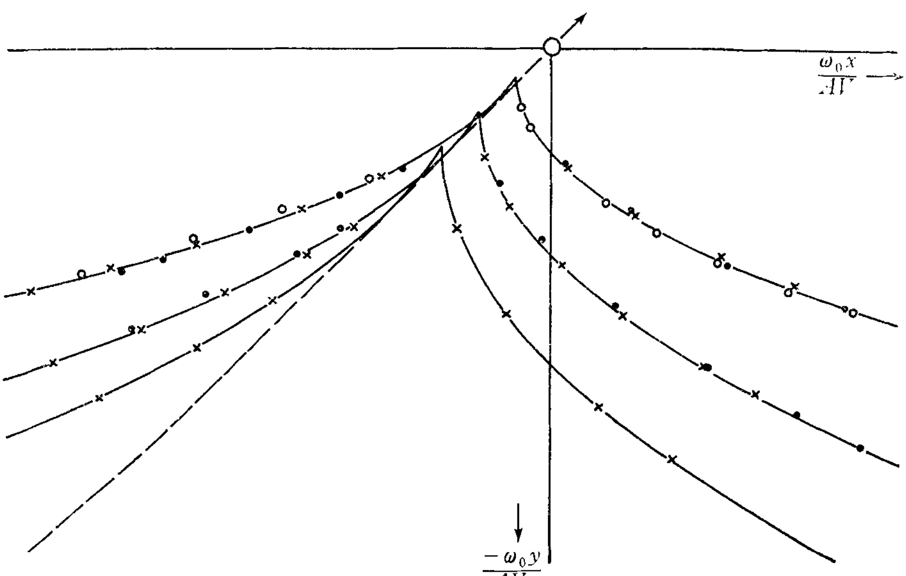

FIGURE 2. Curves of constant phase for $\alpha = 45°$. Experimental points are for the 0·94 cm diameter cylinder moving with velocities of: ●, 0·308 cm/s; ×, 0·314 cm/s; ○, 0·325 cm/s.

scribed by Mowbray (1966). The diameter of the parallel beam of light traversing the tank was 46 cm.

In these experiments the linear density distribution over the depth of the working section was sufficiently close to the exponential distribution implied by the constant Väisälä–Brunt frequency assumed in the theory. An inhomo-

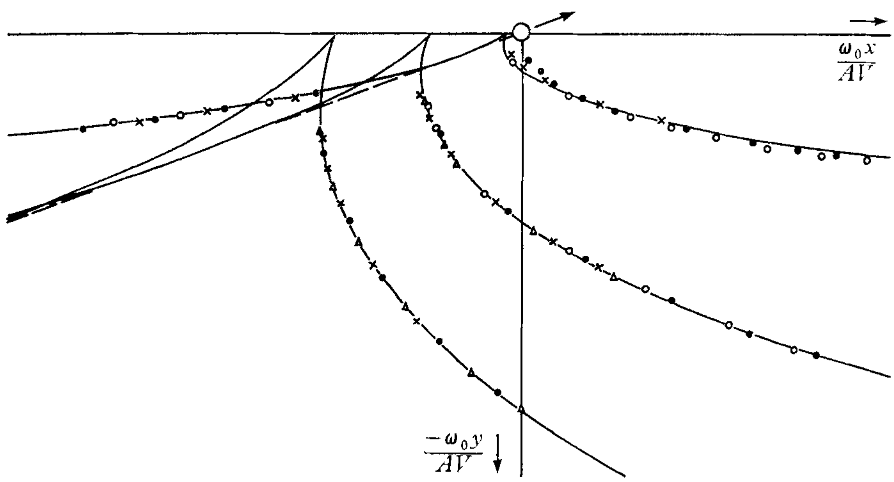

FIGURE 3. Curves of constant phase for $\alpha = 20°$. Experimental points are for the 0·94 cm diameter cylinder moving with velocities of: ○, 0·255 cm/s; ×, 0·346 cm/s; ●, 0·347 cm/s; △, 0·492 cm/s.

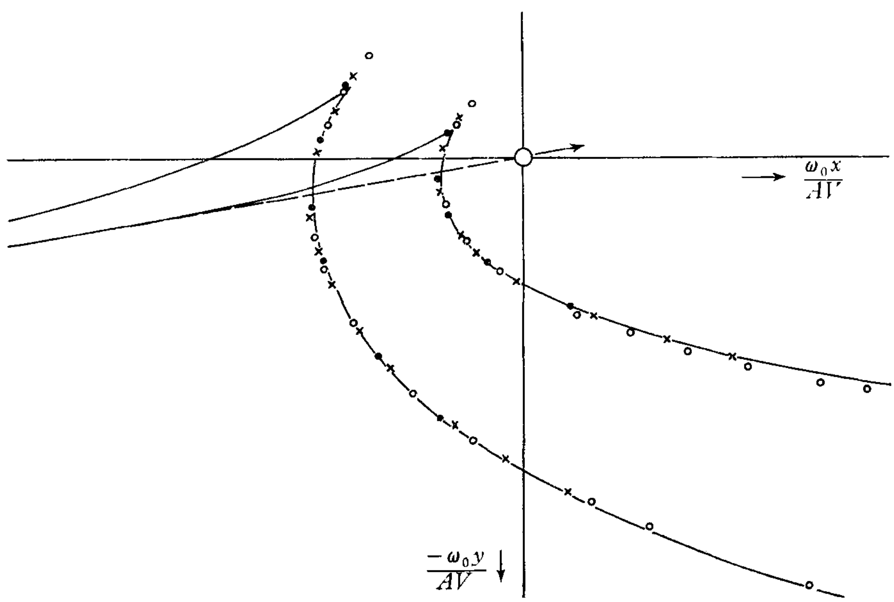

FIGURE 4. Curves of constant phase for $\alpha = 10°$. Experimental points are for the 0·94 cm diameter cylinder moving with velocities of: ×, 0·388 cm/s; ○, 0·442 cm/s; and ●, for the 0·24 cm diameter cylinder moving with a velocity of 0·346 cm/s.

geneity with respect to ω_0 results in a slight bending of the wave crests, but the effect was negligible in the present experiments. The specific gravity of the salt solution varied from 1·082 to 1·023 through the 46 cm working section.

Some photographs of the internal wave systems are shown in figure 6, plate 1 and figure 7*a*, plate 2. The black vertical line on the photographs is the shadow of the model supports and we are looking along the axis of the cylinder. The angle and the direction of motion are shown by the wake from the model. In figures 1–4 the theoretical and experimental phase configurations are compared. When α is 90° and 45° the agreement is very good. At the smaller values

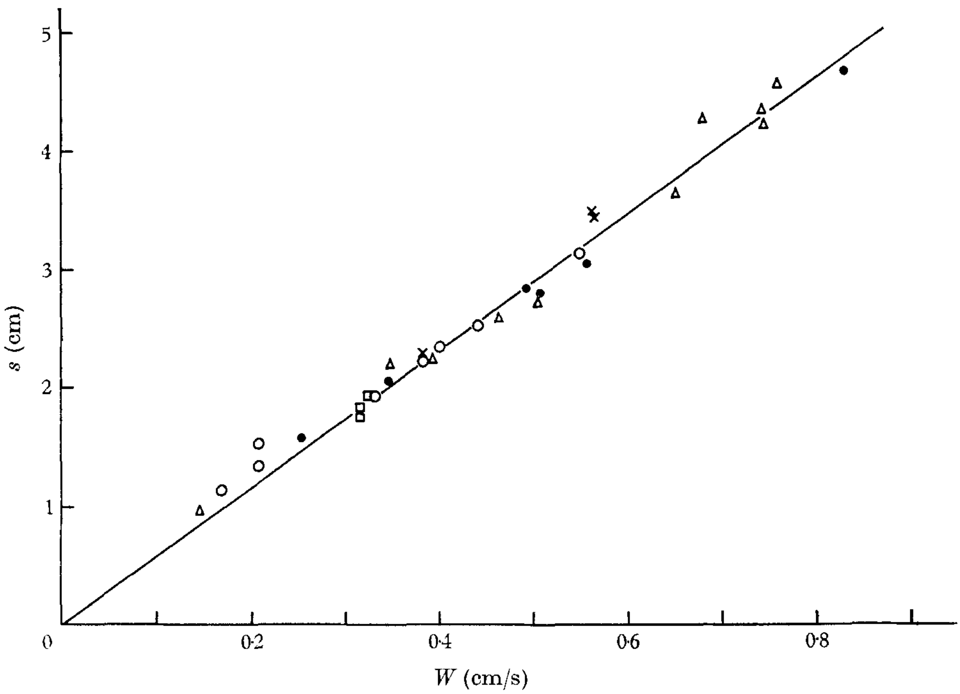

FIGURE 5. The variation in the distance s with the cylinder velocity W. Experimental results are from the 0·94 cm diameter cylinder with: ×, $\alpha = 90$; □, $\alpha = 45°$; ●, $\alpha = 20°$; ○, $\alpha = 10°$; and from the 0·24 cm diameter cylinder with △, $\alpha = 20°$.

of α the agreement is very good outside the wave interaction regions, but inside this region the schlieren pictures are rather confusing. A smaller cylinder, 20 cm long and 0·24 cm diameter, was also used but the photographs were still difficult to analyse. Figure 7*b* shows the interaction region with several diamond shapes present and the lines of constant phase are approximately along the diagonals of these.

When the model moves slowly the waves are close together and there are few waves visible. At higher velocities the wave length is greater and more waves are visible. The variation is such that the ratio of cylinder velocity to the wave spacing in a particular direction from the cylinder is constant. As an example, the co-ordinates at which lines of the same phase cross the path of the body are given by

$$(x, y) = (a + 2\pi n)(V/\omega_0)(\cot\alpha, 1),$$

where n is an integer and a is a constant. The distance, s, between the waves along the path of the body is thus $2\pi W/\omega_0$. As ω_0 was constant in the experiments a plot of s sgainst W, for any cylinder and any angle α, should be a straight line of gradient $2\pi/\omega_0$. The experimental results are presented in figure 5. The straight line has a gradient of 5·83 compared with $2\pi/\omega_0 = 5{\cdot}6$, calculated from the density stratification.

Figures 7c and d show the wave system shortly after an impulsive start from rest to a constant velocity and some of the transient waves are just visible.

4. Conclusions

The experiments have shown that the linearized theory predicts the phase configuration of the internal waves reasonably well and cine films confirm that the wave crests are stationary with respect to the moving cylinder.

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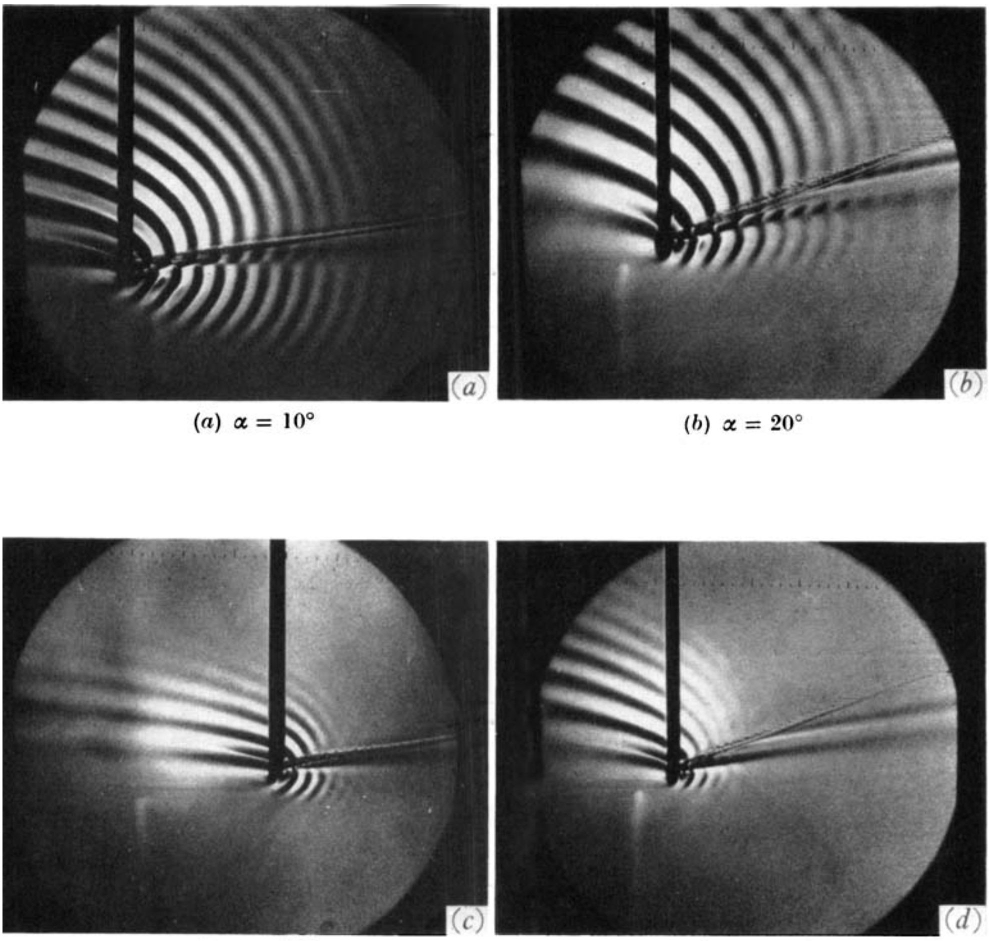
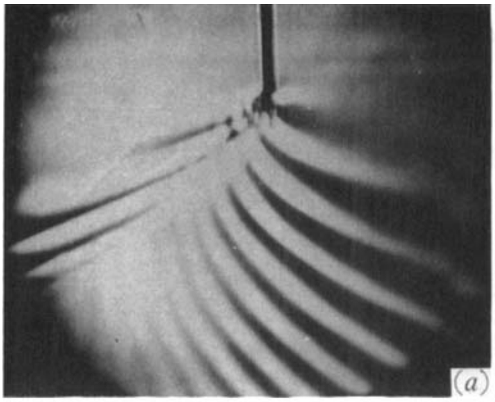


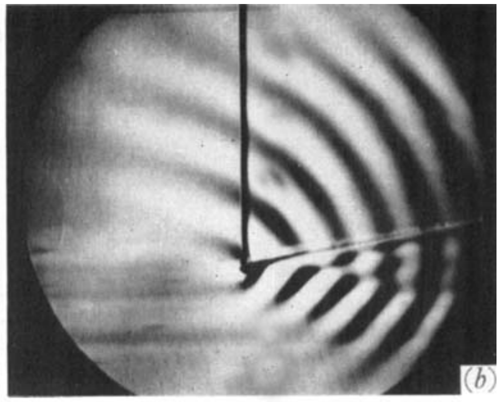
(*a*) $\alpha = 10°$ (*b*) $\alpha = 20°$

(*c*) $\alpha = 10°$ (*d*) $\alpha = 20°$

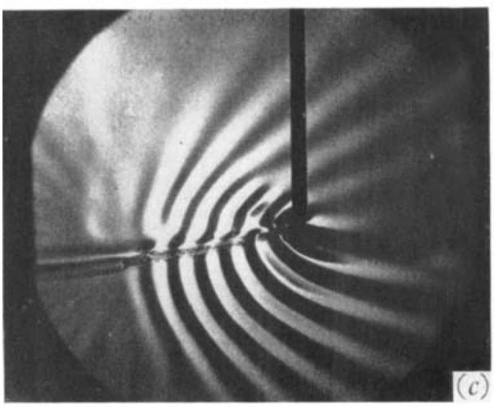
FIGURE 6. The wave system around the 0·94 cm diameter cylinder moving with a constant velocity of (*a*) 0·39 cm/s, (*b*) 0·42 cm/s; (*c*) 0·21 cm/s and (*d*) 0·25 cm/s.



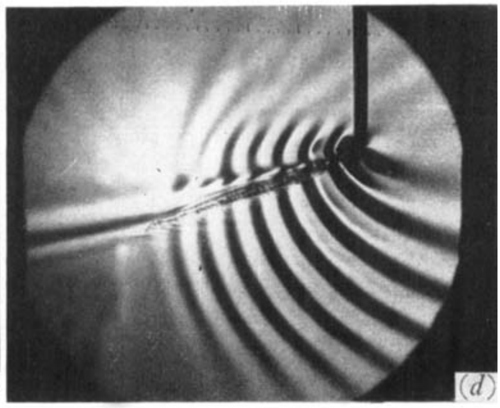
(*a*) $\alpha = 45°$



(*b*) $\alpha = 10°$



(*c*) $\alpha = 10°$



(*d*) $\alpha = 20°$

FIGURE 7. Internal wave patterns (*a*) 0·94 cm diameter cylinder moving with a velocity of 0·32 cm/s. (*b*) 0·24 cm diameter cylinder moving with a velocity of 0·68 cm/s. (*c*) and (*d*) are impulsive starts of the 0·94 cm diameter cylinder to velocities of 0·44 cm/s and 0·49 cm/s respectively.